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ON THE DESCRIPTION OF THE STRESS–STRAIN STATE OF A WEDGE-SHAPED PLATE†

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A refined solution of the Woinowsky-Krieger problem [1, 2] of the deformation of a transversally isotropic wedge-shaped plate is constructed taking the shear strain into account. A Mellin transformation is used.

Since the plate is transversally isotropic and has a free edge [3, 4], it is necessary to apply refined theories to the Woinowsky-Krieger problem (taking the shear strain into account). The cutting forces must then be found more accurately because they can exceed the corresponding forces in the Kirchhoff–Love theory by as much as an order of magnitude in the presence of a free edge.

We shall use the equilibrium equations in terms of the stress components of the three-dimensional problem of the theory of elasticity in orthogonal curvilinear coordinates. By integrating across the shell (with the shear and normal stresses τ_{13}, τ_{23}, σ_{12} and σ_{11}, σ_{22} being constant), we obtain the equilibrium equations (the zero approximation, Vekua's moment-free case [5])

$$\frac{\partial}{\partial\alpha}(BN_1^0)-N_2^0\frac{\partial B}{\partial\alpha}+\frac{\partial}{\partial\beta}(AS_{12}^0)+S_{21}^0\frac{\partial A}{\partial\beta}+Q_1^0k_1AB+ABF_1=0 \quad (1\to 2, A\to B)$$

$$-(k_1N_1^0+k_2N_2^0)+\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BQ_1^0)+\frac{\partial}{\partial\beta}(AQ_2^0)\right]+F=0 \qquad (1)$$

(α and β are the orthogonal curvilinear coordinates chosen on the median surface of the shell, N_1^0, N_2^0 and $S_{12}^0=S_{21}^0=S^0$ are the normal and shear forces arising in the shell, Q_1^0, Q_2^0 are the cutting forces, F_1, F_2, F are the projections of load intensity onto the α, β, z directions, k_1 and k_2 are the curvatures of the coordinate lines α and β, and A, B are the coefficients of the first quadratic form of the median surface of the shell).

The presence of cutting forces in the equilibrium equations (1) indicates that the transverse shear is taken into account in these equations [6–8]. The application of Eqs (1) develops and generalizes Timoshenko's shear model [9].

The study of the transverse shear in shells in the case under consideration has been distinguished as an independent problem governed by a system of differential equations of lower order than in the original problem.

The analysis of the stress–strain states of anisotropic shells reveals [10, 11] that the strain due to the transverse shear predominates over those due to the normal stresses and compression.

From physical considerations, we have

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$$\varepsilon_{j3} = \varepsilon^0_{j3} \quad (j = 1, 2) \tag{2}$$

(ε_{j3} and ε^0_{j3} are the strain components of the transverse shear determined from Timoshenko's theory [9] and from the zero approximation, respectively).

We set the physical condition

$$U_j = U^*_j + U^0_j, \qquad w = w^* + w^0 \tag{3}$$

(U_1, U_2, w are the projections of the displacement vector of points on the median surface in the directions α, β, z, U^*_1, U^*_2, w^* are the projections of the displacement vector of points on the median surface determined from the classical theory (using Kirchhoff's hypothesis), and U^0_1, U^0_2, w^0 are the projections of the displacement vector of points on the median surface determined by the method under consideration).

The expressions for the strain components ε_{j3} and ε^0_{j3}, taking (2) and (3) into account, lead to the relation

$$\gamma_1 = -A^{-1}\partial w_*/\partial\alpha + k_1 U^*_1 \quad (1 \to 2, A \to B) \tag{4}$$

(γ_1 and γ_2 are the angles of rotation of the normal to the median surface).

Conditions (4) and, consequently, (2) are consistent with the condition of equivalence [12]. It follows that by considering the problem separately by the method in question and the classical theory one can determine the solution of the original problem. Five equilibrium equations are used to construct the complete stress–strain state of the shell. To state the boundary conditions one must compute the work done by the forces acting along the contour of the shell given the appropriate displacements.

Setting $k_1 = k_2 = 0$ in (1), we obtain the equilibrium equations for plates corresponding to the method under consideration. The first two equations define the generalized plane strain state of the plate. The third equilibrium equation

$$\partial Q^0_1/\partial x + \partial Q^0_2/\partial y + F = 0 \tag{5}$$

defines the transverse shear [6–8] (x and y are the Cartesian coordinates of points in the median plane of the plate).

The substitution of the expressions for the cutting forces into (5) leads to the equation

$$\Delta w^0 = -F/2G'kh \tag{6}$$

for the bending of transversally-isotropic plates (Δ is the Laplace operator G' is the shear modulus in the normal plane, and k is the shear coefficient, defined by the law governing the variation of shear stresses across the plate thickness [12]). We remark that Eq. (6) can also be obtained from the principle of virtual displacement. The equation $\Delta w^0 = 0$ is used in what follows.

We consider a wedge-shaped plate (see Fig. 1) and assume that the edge $\theta = 0$ is fixed, while the edge $\theta = \alpha_1$ is free, except for the point $r = r_0$, where a concentrated force P is applied.

The solution of this problem is known in the classical setting [1, 2]. We will construct a solution in which the transverse shear is taken into account. The function $f(r)$ can be represented using Mellin's formula as follows:

$$f(r) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} r^{-s}ds\int_0^\infty \rho^{s-1}f(\rho)d\rho \tag{7}$$

(s is a parameter and σ is a real constant that satisfies certain restricting conditions).

In particular, for the force P concentrated at $r = r_0$ we get

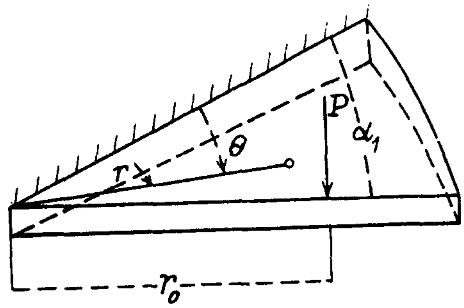

Fig. 1.

$$f(r)=\frac{P}{2\pi i r_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{r}{r_0}\right)^{-(s+3)}ds \tag{8}$$

For the deflection W^0 we find from the equation $\Delta W^0=0$ that

$$W^0 = r^{-s-2}(C(s)\cos((s+2)\theta) + D(s)\sin((s+2)\theta))$$

($C(s)$ and $D(s)$ are the functions to be found).

Using the boundary conditions $w^0|_L=0$ and $\theta=0$, where L is the rectilinear fixed edge (see Fig. 1), we get $C(s)=0$.

From the boundary conditions at the free edge of the plate we have $(Q_t^0)_{\theta=\alpha_1}=f(r)$. (Formula (8) can be used to determine $f(r)$; Q_t^0 is the tangential shear force.) Then

$$D(s)=\frac{Pr_0^{s+2}}{2hG'(s+2)\cos((s+2)\alpha_1)k}$$

After some reduction we get [13]

$$w^0=\frac{P}{4hG'\pi k}\ln\left[\frac{\operatorname{ch}\xi+\sin\eta}{\operatorname{ch}\xi-\sin\eta}\right],$$

$$\xi=\frac{\pi}{2\alpha_1}\ln\frac{r_0}{r},\quad \eta=\frac{\pi\theta}{2\alpha_1} \tag{9}$$

for the desired deflection.

For $r=r_0$ and $\theta=\alpha$ (the free edge) we have a logarithmic singularity for the deflection w^0 of the plate. $w^0\to 0$ as $G'\to\infty$ (the classical case).

The physical equation for the cutting forces leads to the expression

$$Q_t^0=\frac{P}{2r\alpha_1}\frac{\operatorname{ch}\xi\cos\eta}{\operatorname{ch}^2\xi-\sin^2\eta} \tag{10}$$

The total deflection can be determined from (3). Compared with the classical argument, it is a more rational choice to use (10) to construct the diagrams of the shear forces. It can also lead to much more accurate results [3, 4]. In this case the natural boundary conditions are satisfied at the plate edge.

The results of computing the deflections for ν equal to zero and $\alpha_1=\pi/4$ are presented in Table 1 (ν is Poisson's ratio and E is Young's modulus). In particular, it follows from Table 1 that w/w^* is equal to 2.1759 for $2h/r_0=0.4$, $E/G'=8$, and $r/r_0=0.9$.

Table 1

$2h/r_0$	r/r_0	$z' = E/G'$	$(w^0 : Pr_0^2/D) \cdot 10^{+3}$	w^0/w^*
	0,1		0,0031832	
0,05	0,2		0,012739	
	0,3		0,028726	
	0,4		0,051371	
	0,5	2	0.081300	
	0,6		0,11997	
	0,7		0,17063	
	0,8		0,24133	
	0,9		0,55875	
0.35	0,1		0,15597	
	0,2		0,62422	
	0,3		1,4076	
	0.4		2,5172	
	0,5		3,9837	
	0.6		5.8783	
	0.7		8,3610	
	0,8		11,825	0,16904
	0,9		17,579	0,22507
0,05	0,1	8	0,012733	
	0,2		0,050957	
	0,3		0.1149	
	0,4		0,20549	
	0,5		0,32520	
	0,6		0,47987	
	0,7		0,68253	
	0,8		0,96534	
	0,9		1.4350	
0,25	0,1		0,31832	
	0,2	8	1,2739	
	0,3		2,8726	
	0,4		5,1371	
	0,5		8,4300	
	0,6		11,997	
	0,7		17,063	
	0,8		24,133	0,34498
	0,9		35,875	0.45932
0,3	0,1		0,45837	
	0,2		1,8344	
	0,3		4.1365	
	0.4		7,3974	
	0,5		11,707	
	0.6		17,275	
	0,7		24,571	
	0,8		34,752	0,49678
	0,9		51,659	0,66141
0,35	0.1		0,62390	
	0,2		2,4969	
	0,3		6,6302	
	0,4		10,069	
	0,5		15,935	
	0,6		23,513	
	0.7		33,444	
	0,8		47,301	0,67617
	0,9		70,314	0,90026
0,4	0,1		0,81489	
	0,2		3,2612	
	0,3		7,3537	
	0,4		13,151	
	0,5		20,81	
	0,6		30,711	
	0,7	8	43,682	
	0,8		61,781	0.88316
	0,9		91,839	1.1759

The ratio w/w^* is equal to 1.88316 if $2h/r_0=0.4$, $E/G'=8$, and $r/r_0=0.8$. The results indicate that the transverse shear has a significant effect on the deflection of the plate. Because of this it is necessary to refine the problem under consideration. We observe that $w^0 w^* \sim E/G'$ and $w^0/w^* = 1 + w/w^*$.

This enables us to use the results presented in Table 1, e.g. for $E/G'=20$; 40; 60, we can easily find that the transverse shear has a stronger effect on the deflection of the plate. For $2h/r_0=0.4$, $E/G'=40$, and $r/r_0=0.9$ the ratio w/w^* is equal to 6.8795. For comparison (see Table 1), note that w^* is equal to $0.0815Pr_0^2/D$ for $r=r_0$ [1, 9].

The comparisons which have been made (see Table 1) show that the effect of transverse shear for a plate is governed by the plate thickness and the ratio E/G', which is equal to zero in the classical formulation of the problem ($G'\to\infty$), i.e. the classical theory is not affected at all by variations of E/G'.

By applying the correspondence principle [14], one can extend the results to viscoelastic plates. The results are transformed from the elastic problem to the viscoelastic one as follows:

$$P/D \to \int_0^t \Pi(t-s)dP(s) + (2B_0)^{-1}\int_0^t g_{12}(t-s)dP(s) \tag{11}$$

$$P/G' \to \int_0^t \Pi_0(t-s)dP(s), \quad P=P(t) \tag{12}$$

(B_0, $\Pi(t)$, $g_{12}(t)$ are quantities determined experimentally [14], B_0 is the bulk modulus of elasticity, $\Pi(\tau)$ is the creep function, $g_{12}(t)$ is a function whose values are determined by Poisson's ratio, $\Pi_0(t)$ is the creep function determined by the transverse shear, t is the time; and s is a parameter).

No refinement of the problem under consideration from the viewpoint of Timoshenko's direct shear model has been obtained to date.

The case of tangential and normal stresses uniformly distributed across the plate, which corresponds to considering the transverse shear separately [6–8], follows directly from Timoshenko's shear model as $z\to 0$ (the median surface of the shell is considered).

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